Examples From

Solving

Real Roots

Polynomial Equations.

(5 New Theorems And Many New Definitions.)

(New and Simple Methods of Solving Polynomial Equations.(All Real Roots.) PART – 2.)

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Examples From ‘Solving Real Roots Polynomial Equations’.

These examples are taken from my book **‘Solving Real Roots Polynomial Equation’(New And Simple Methods Of Solving Polynomial Equations.(All Real Roots.) Part -2.).**

Before studying these examples, please see **‘New And Simple Methods Of Solving Polynomial Equations Part- 1.’** otherwise it is difficult to understand.(To read, go to www.rkmath.yolasite.com.)

Theorem 11.

In an equation, $\mathbf{X^n+Ax^{n-1}+Bx^{n-2}}$- - - - - - - - - - - - - - $\mathbf{YX^2+ZX+K}$ = 0, whose **all op-roots are real and positive**. A, B, Y and Z are the co-efficient of X^{n-1}, X^{n-2}, X^2 and X respectively and K is a constant. And n is the highest power of X. (n = Positive whole number & > 1.)

The smallest op-root (rs) of the above equation is more than or equal to $\frac{K}{Z} + (n-1)\frac{K}{Z}(rsRL)$ and less than **2** $\left[\frac{K}{Z} + (n-1)\frac{K}{Z}(rsRL) \right]$ **,** Where rsRL = D and D value can be calculated by the formula

$$\frac{\frac{(n-1)(n-2)D^2}{2} + (n-1)D}{[(n-1)D+1]^2} = \frac{YK}{Z^2}$$

.(The smallest op-root (rs) = more precisely, the op-root which is not bigger than any other op-root in the equation.)

rs $\geq \frac{K}{Z} + (n-1)\frac{K}{Z}(rsRL)$

& **rs** $< 2\left[\frac{K}{Z} + (n-1)\frac{K}{Z}(rsRL)\right]$

Solving Equations Using Theorem 11.

Example 1.1

$X^4+45X^3+729X^2+4955X+11550=0$ -------------- equation(1)

As per theorem 11

The smallest op-root = rs

rs $\geq \frac{K}{Z} + (n-1)\frac{K}{Z}(rsRL)$

Calculating rsRL

$$\frac{\frac{(n-1)(n-2)D^2}{2}+(n-1)D}{[(n-1)D+1]^2} = \frac{YK}{Z^2}$$

n = 4 y = 729 z = 4955 k = 11550

$$\frac{3D^2+3D}{9D^2+6D+1} = \frac{729\times11550}{4955^2} = 0.3429$$

D = 0.3769

rsRL = D = 0.3769

rs $\geq \frac{K}{Z} + (n-1)\frac{K}{Z}(rsRL)$

rs$\geq \frac{11550}{4955} + 3 \times \frac{11550}{4955} \times 0.3769$

$\geq$ **4.967**

The smallest op-root rs $\geq$ **4.967 and (rs** $<$ **4.967** $\times$ **2** $=$ **9.934)**

Take less immediate whole number of 4.967 ie 4.

Diminish the op-roots of equation(1) by 4

4	1	45	729	4955	11550
	0	4	164	2260	10780
	1	41	565	2695	770
	0	4	148	1668	
	1	37	417	1027	
	0	4	132		
	1	33	285		
	0	4			
	1	29			

The new equation is

$X^4+29X^3+285X^2+1027X+770=0$ equation (2)

<u>Calculating rsRL of equation (2).</u>

$$\frac{\frac{(n-1)(n-2)D^2}{2}+(n-1)D}{[(n-1)D+1]^2}=\frac{YK}{Z^2}$$

n = 4 y = 285 z = 1027 k = 770

$$\frac{3D^2+3D}{9D^2+6D+1} = \frac{285\times770}{1027^2} = 0.2080$$

D = 0.1108

rsRL = D = 0.1108

$$rs \geq \frac{K}{Z} + (n-1)\frac{K}{Z}(rsRL)$$

$$rs \geq \frac{770}{1027} + 3 \times \frac{770}{1027} \times 0.1108$$

$$\geq 0.9991$$

The smallest op-root rs $\geq$ 0.9991 and (rs $<$ 0.9991$\times$ 2 $=$ 1.9982)

Assuming this op-root may be an integer, diminish the op-roots of equation(2) by 1.(because the whole number between these values is 1.)

1	1	29	285	1027	770
	0	1	28	257	770
	1	28	257	770	0
	0	1	27	230	
	1	27	230	540	
	0	1	26		
	1	26	204		
	0	1			
	1	25			

The smallest op-root rs of equation (1) = 4+1 = 5

The new equation is

$X^3+25X^2+204X+540=0$ -------------------- equation (3)

The other op-roots can be found in the same way.

The op-roots of equation(1). are 5,11,14 and 15.

The roots of equation(1) are -5,-11,-14 and -15.

<u>Example 1.2</u>

$X^3-7x-6=0$ ------------------ equation(1)

This equation has negative op-root(s) too. So this equation should be changed to all positive op-roots equation.

As per **theorem 3,(See 'New And Simple Methods Of Solving Polynomial Equations- Part 1')** .(To read go to www.rkmath.yolasite.com.)

The smallest op-root = rs

$$\text{rs} \geq \frac{A-(n-1)\sqrt{A^2-TB}}{n}$$

A = 0 B = -7 n = 3

The smallest op-root rs $\geq \frac{0-2\sqrt{0^2-3\times(-7)}}{3}$ **= --3.0550**

Increase the op-roots of equation(1) by 4.

4	1	0	--7	--6
	0	4	16	36
	1	4	9	30
	0	4	32	
	1	8	41	
	0	4		
	1	12		

The new equation is

$X^3+12X^2+41X+30=0$ ----------- equation (2)

As per theorem 11,

The smallest op-root = rs

$$\textbf{rs} \geq \frac{K}{Z} + (n-1)\frac{K}{Z}(rsRL)$$

Calculating rsRL of equation(2).

$$\frac{\frac{(n-1)(n-2)D^2}{2} + (n-1)D}{[(n-1)D+1]^2} = \frac{YK}{Z^2}$$

n = 3 y = 12 z = 41 k =30

$$\frac{D^2+2D}{4D^2+4D+1} = \frac{12\times30}{41^2} = 0.2141$$

D = 0.1831

rsRL = D = 0.1831

rs $\geq \frac{K}{Z} + (n-1)\frac{K}{Z}(rsRL)$

rs $\geq \mathbf{\frac{30}{41} + 2 \times \frac{30}{41} \times 0.1831}$

$\geq$ **0.9996**

The smallest op-root rs of equation(2) $\geq$ **0.9996** **and (rs** $<$ **0.9996**$\times$ **2** $=$ **1.9993)**

Assuming this op-root may be an integer, diminish the op-roots of equation(2) by 1.(because the whole number between these values is 1.)

1	1	12	41	30	
	0	1	11	30	
	1	11	30		0
	0	1	10		
	1	10	20		
	0	1			
	1	9			

The smallest op-root rs of equation(2) = 1

The new equation is

$X^2+9X+20=0$ equation(3)

The op-roots of equation(3) are 4 and 5.

The op-roots of equation(2) are 1 , (1+4)=5, (1+5)=6 .

The op-roots of equation (1) are (1 – 4 =) -3, (5 - 4) = 1, (6 - 4) = 2.

The roots of equation(1) are 3, -1 and -2.

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One Time Squaring

By changing the coefficients of original equation(equation 1) ,we can make new equation (equation 2) whose op-roots are the square of the original equation's op-roots (equation 1's op-roots). This idea has been used in the following example. Here equation's op-roots are squared only one time.

Example 2.1

$X^3+145X^2+6216X+77220=0$ ……….. equation (1)

(op-root = a ,b …….)

$\mathbf{X}^3+8593\mathbf{X}^2+1.6244\times 10^7\mathbf{X}+5.9629\times 10^9=0$ …………. equation (2)

(op-root = a^2,b^2…….)

As per theorem 11,

The smallest op-root = rs

rs $\geq \ \frac{K}{Z} + (n-1)\frac{K}{Z}(rsRL)$

Calculating rsRL of equation(2)

$$\frac{\frac{(n-1)(n-2)D^2}{2} + (n-1)D}{[(n-1)D+1]^2} = \frac{YK}{Z^2}$$

n = 3 y = 8593 z = 1.6244$\times 10^7$ K = 5.9629$\times 10^9$

$$\frac{D^2+2D}{4D^2+4D+1} = \frac{8593\times 5.9629\times 10^9}{(1.6244\times 10^7)^2} = 0.1941$$

D = 0.1543

rsRL = D = 0.1543

rs $\geq \ \frac{K}{Z} + (n-1)\frac{K}{Z}(rsRL)$

rs $\geq \ \frac{5.9626\times 10^9}{1.6244\times 10^7} + 2\times\frac{5.9626\times 10^9}{1.6244\times 10^7}\times 0.1543$

rs $\geq$ 480.3918

rs of equation (2) $\geq$ 480.3918

rs of equation (1) $\geq$ $(480.3918)^{\frac{1}{2}}$

rs of equation (1) $\geq$ 21.9178 and (rs of equation(1) $<$ 21.9178 $\times 2^{\frac{1}{2}} =$ 30.9965)

Take immediate less whole number of 21.9178 ie 21.

Diminish the op-roots of equation(1) by 21.

21	1	145	6216	77220
	0	21	2604	75852
	1	124	3612	1368
	0	21	2163	
	1	103	1449	
	0	21		
	1	82		

The new equation is

$X^3+82X^2+1449X+1368=0$ ------------------ equation (3).

$\mathbf{X}^{\mathbf{3}}+3826\mathbf{X}^{\mathbf{2}}+1.8752 \times 10^6\mathbf{X}+1.8714\times 10^6=0$ --------- equation (4).

Calculating rsRL of equation (4)

$$\frac{\frac{(n-1)(n-2)D^2}{2}+(n-1)D}{[(n-1)D+1]^2} = \frac{YK}{Z^2}$$

n = 3 y = 3826 z = 1.8752 $\times 10^6$ K = 1.8714$\times 10^6$

$$\frac{3D^2+3D}{9D^2+6D+1} = \frac{3826\times1.8714\times10^6}{(1.8752\times10^6)^2} = \mathbf{0.0020}$$

D = 0.0010

rsRL = D = 0.0010

$$\text{rs} \geq \frac{K}{Z} + (n-1)\frac{K}{Z}(rsRL)$$

$$\text{rs} \geq \frac{1.8714\times10^6}{1.8752\times10^6} + 2\times\frac{1.8714\times10^6}{1.8752\times10^6}\times 0.0010$$

rs of equation (4) $\geq$ 0.9999

rs of equation (3) $\geq$ $(0.9999)^{\frac{1}{2}}$

rs of equation (3) $\geq$ 0.9999 and (rs of equation (3) $<$ 0.9999 $\times 2^{\frac{1}{2}}$ $=$ 1.4142)

Assuming this op-root may be an integer, diminish the op- roots of equation (3) by 1.(because the whole number between these values is 1).

1	1	82	1449	1368
	0	1	81	1368
	1	81	1368	0
	0	1	80	
	1	80	1288	
	0	1		
	1	79		

The smallest op-root rs of equation (1) = 21+1 = 22

The new equation is

$X^2+79X+1288=0$ equation(5)

The op-roots of equation (5) are 23 and 56.

The op-roots of equation (1) are,

22

23+22 = 45

56 + 22 = 78

The roots of equation (1) are – 22, -- 45 and -- 78.

Four Time Squaring

(It Seems To Be Nice)

In the last example, the op-roots are squared one time.(It can be done by using coefficients). Here op-roots are squared four times continuously.

Example 3.1

$X^3+52X^2+885X+4950=0$ ------------------------- equation (1)

(op-root = a ,b …….)

$\mathbf{X^3}+934\mathbf{X^2}+2.6842\times 10^5\mathbf{X}+2.4502\times 10^7=0$ ------ equation(2)

(op-root = a^2,b^2…….)

$\mathbf{X^3}+3.3550\times 10^5\mathbf{X^2}+2.6281\times 10^{10}\mathbf{X}+6.0037\times 10^{14}=0$ equation(3)

(op-root = a^4 ,b^4 …….)

$\mathbf{X^3}+6.0001\times 10^{10}\mathbf{X^2}+2.8785\times 10^{20}\mathbf{X}+3.6044\times 10^{29}=0$ equation(4)

(op-root = a^8,b^8…….)

$\mathbf{X^3}+3.0244\times 10^{21}\mathbf{X^2}+3.9602\times 10^{40}\mathbf{X}+1.2992\times 10^{59}=0$ ---- equation(5)

(op-root = a^{16},b^{16} …….)

As per theorem 11,

The smallest op-root = rs

$$\mathbf{rs} \geq \frac{\boldsymbol{K}}{\boldsymbol{Z}}+(\boldsymbol{n}-\mathbf{1})\frac{\boldsymbol{K}}{\boldsymbol{Z}}(\boldsymbol{rsRL})$$

Calculating rsRL of equation (5)

$$\frac{\frac{(n-1)(n-2)D^2}{2}+(n-1)D}{[(n-1)D+1]^2} = \frac{YK}{Z^2}$$

n = 3 y = 3.0244 $\times 10^{21}$ **Z = 3.9602** $\times 10^{40}$

K = 1.2992 $\times 10^{59}$

$$\frac{D^2+2D}{4D^2+4D+1} = \frac{3.0244\times10^{21}\times 1.2992\times10^{59}}{(3.9602 \times10^{40})^2} = 0.2505$$

D = 0.2512

rsRL = D = 0.2512

$$\text{rs} \geq \frac{K}{Z} + (n-1)\frac{K}{Z}(rsRL)$$

$$\text{rs} \geq \frac{1.299\times10^{59}}{3.9602 \times10^{40}} + 2 \times \frac{1.299\times10^{59}}{3.9602 \times10^{40}} \times 0.2512$$

$\geq$ **4.9289** $\times 10^{18}$

The smallest op-root rs of equation (5) $\geq$ **4.9289** $\times 10^{18}$

The smallest op-root rs of equation (1) $\geq$ $(4.9289 \times 10^{18})^{\frac{1}{16}}$

The smallest op-root rs of equation(1) $\geq$ **14.7332 and (rs of equation (1)** $<$ **14.7332**$\times 2^{\frac{1}{16}} =$ **15.3854)**

Assuming this op-root may be an integer, diminish the op-roots of equation(1) by 15.(because the whole number between these values is 15).

15	1	52	885	4950
	0	15	555	4950
	1	37	330	0
	0	15	330	
	1	22	0	
	0	15		
	1	7		

So the op- roots are 15, 15 and (15+7) =22.

The roots are --15, -- 15 and --22

Continuously Squaring

(A straight method)

If you don't like to use synthetic division , you may use this method. The op-roots are continuously squared up to the required level. If the required op-root value is single digit, the op-roots should be squared four times continuously.(It can be done by using coefficients). If the required op-root value is double digit, the op-roots should be squared eight times continuously and so on. (Refer **Graeffe's method** and see the similarities and the differences.)

Example 4.1

$X^3+62X^2+1211X+7150=0$ --------------- equation(1)

(op-root = a ,b …….)

$\mathbf{X^3}+ 1{,}422\mathbf{X^2}+5.7992\times 10^5\mathbf{X}+5.1122\times 10^7=0$ equation(2)

(op-root = a^2,b^2…….)

$\mathbf{X^3}+8.6224\times 10^5\mathbf{X^2}+1.9091\times 10^{11}\mathbf{X}+2.6135\times 10^{15}=0$ equation(3)

(op-root = a^4,b^4 …….)

$\mathbf{X^3}+3.6162\times 10^{11}\mathbf{X^2}+ 3.1941 \times 10^{22}\mathbf{X}+6.8304\times 10^{30}=0$ equation(4)

(op-root = a^8,b^8…….)

$\mathbf{X^3}+6.6891\times 10^{22}\mathbf{X^2}+1.0153\times 10^{45}\mathbf{X}+4.6654\times 10^{61}=0$ equation(5)

(op-root = a^{16} ,b^{16} …….)

Don't let the coefficients become very big .

Divide the op-roots of equation(5) by 10^{20}

$\mathbf{X^3}+6.6891\times 10^2\mathbf{X^2}+1.0153\times 10^5\mathbf{X}+4.6654\times 10^1=0$ equation(6)

(Now op-root = a_1,b_1……..)

$\mathbf{X^3}+2.4438\times 10^5\mathbf{X^2}+1.0309\times 10^{10}\mathbf{X}+2.1760\times 10^3=0$ equation(7)

(Now op-root = a_1^2, b_1^2........)

$\mathbf{X^3}+3.9104\times 10^{10}\mathbf{X^2}+1.0628\times 10^{20}\mathbf{X}+4.7379\times 10^{6}=0$ equation(8)

(Now op- root = a_1^4, b_1^4........)

$\mathbf{X^3}+1.3165\times 10^{21}\mathbf{X^2}+1.1295\times 10^{40}\mathbf{X}+2.2447\times 10^{13}=0$ equation(9)

(Now op-root = a_1^8, b_1^8........)

$\mathbf{X^3}+1.7107\times 10^{42}\mathbf{X^2}+1.2759\times 10^{80}\mathbf{X}+5.0390\times 10^{26}=0$ equation(10)

(Now op- root = a_1^{16}, b_1^{16}........)

As per theorem 11,

rs $\geq \frac{K}{Z}+(n-1)\frac{K}{Z}(rsRL)$

Calculating rsRL of equation (10)

$$\frac{\frac{(n-1)(n-2)D^2}{2}+(n-1)D}{[(n-1)D+1]^2}=\frac{YK}{Z^2}$$

n = 3 **y = 1.7107×10^{42}** **Z = 1.2759×10^{80}**

K = 5.0390×10^{26}

$$\frac{D^2+2D}{4D^2+4D+1} = \frac{1.7107\times10^{42}\times 5.039\times10^{26}}{(1.2759\times10^{80}\ \ )^2} = 5.2954\times 10^{-92}$$

D = 0

rsRL = D = 0

rs $\geq \frac{K}{Z} + (n-1)\frac{K}{Z}(rsRL)$

rs $\geq \frac{5.0390\times10^{26}}{1.2759\times10^{80}} + 2 \times \frac{5.0390\times10^{26}}{1.2759\times10^{80}} \times 0$

rs $\geq$ **3.9493**$\times 10^{-54}$

The smallest op-root rs of equation (10) $\geq$ **3.9493**$\times 10^{-54}$

The smallest op-root rs of equation (6) $\geq (3.9493 \times 10^{-54})^{\frac{1}{16}}$

The smallest op-root rs of equation (6) $\geq$ **4.5949**$\times 10^{-4}$

The smallest op-root rs of equation (5) $\geq (4.5949 \times 10^{-4})10^{20}$

The smallest op-root rs of equation (5) $\geq$ **4.5949**$\times 10^{16}$

The smallest op-root rs of equation (1) $\geq \quad (\mathbf{4.5949 \times 10^{16}})^{\frac{1}{16}}$

The smallest op-root rs of equation (1) $=$ **11 (corrected to two numbers)**

verifying

11	1	62	1211	7150
	0	11	561	7150
	1	51	650	0
	0	11	440	
	1	40	210	
	0	11		
	1	29		

The smallest op-root rs = 11

The new equation is

$X^2+29X+210=0$ -------------------------- equation (11)

The op- roots of **equation (11) are 14 and 15**

The op- roots of **equation(1) are 11, 25(11+14) and 26 (11+15)**

The roots of **equation (1) are --11, --25 and --26 .**

Convergence

In first method (example1.1 &1.2),in every step, the value of $\frac{K}{Z}+(n-1)\frac{K}{Z}(rsRL)$ is more than **50%** of the smallest op-root(rs). Sometimes it is **100 %** correct value, in first step itself.

In second method (example2.1),in every step, the value of $\frac{K}{Z}+(n-1)\frac{K}{Z}(rsRL)$ is more than **70%** of the smallest op-root(rs). Sometimes it is 100 % correct value, in first step itself.

In third method (example3.1),in every step, the value of $\frac{K}{Z}+(n-1)\frac{K}{Z}(rsRL)$ is more than **90%** of the smallest op-root(rs). Sometimes it is **100 %** correct value, in first step itself.

Though I didn't show any example ,you can solve the equations which op-roots or coefficients or constant are irrational or fraction or decimal. In these methods you get answer in numerical value (approximate). But I have shown how to change it to exact answer(symbolic form)(To know more, go to www.rkmath.yolasite.com)

There are a lot of methods available for solving polynomial equations. The following are few methods **Newton's method, Secant method, Laguerre's method, Householder's method, Halley's method, Graeffe's method and Brent's method .** Refer these and other methods to get more ideas, and, from the math books and the web sites.

Simple formula

For small equations, you need not to use the above methods. You may use the following simple formula.

$$\text{rs} > \sqrt{\frac{k^2}{z^2 - 2yk}} \quad \text{and} \quad \text{rs} \leq \sqrt{\frac{nk^2}{z^2 - 2yk}}$$

(This formula can also be applied to all negative op- roots equation and to positive and negative op- roots equation with care. Since the signs are to be changed accordingly.)

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